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Quantization of the Gaudin System

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Abstract

In this article we exploit the known commutative family in $Y(\mathfrak{gl}_n)$ - the Bethe subalgebra - and its special limit to construct quantization of the Gaudin integrable system. We give explicit expressions for quantum hamiltonians $QI_k(u),\ k = 1,\ldots,n$. At small order $k = 1,\ldots,3$ they coincide with the quasiclassic ones, even in the case $k = 4$ we obtain quantum correction.

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1 Introduction

Quantization. The quantization problem has different aspects and different approaches therein. The quantization consists in constructing a deformation of a Poisson algebra of functions on a phase manifold and an appropriate representation. We are interested in the first part of quantization problem in a specific case of integrable systems. An integrable system is a hamiltonian system on a Poisson manifold (M^{2n}, π) where π is a Poisson bivector, given by a hamiltonian H such that there exists a family of algebraically independent functions $I_1, \ldots, I_n$ in involution $\{I_i, I_j\} = 0$ and $H = I_1$. In this language an integrable system is a subalgebra $\Im \subset C(M)$ which is also commutative subject to the Poisson bracket and is generated by the appropriate number of generators. This property of being appropriate is even more ambiguous at the quantum invariant level, so we do not discuss it here.

Traditionally the quantization of a Poisson algebra on $C(M)$ is an associative algebra $A(M)$ isomorphic as a linear space to the space of formal series $C(M)[\hbar]$ with a fixed isomorphism of linear spaces

$$q : C(M)[\hbar] \to A(M)$$

called a quantization correspondence, such that

$$q(f) * q(g) = q(fg) + O(\hbar) \quad \text{and} \quad q(f) * q(g) - q(g) * q(f) = \hbar q(\{f, g\}) + O(\hbar^2)$$

where f, g are functions on M, $*$ is the multiplication on $A(M)$ and $\{f, g\}$ is the Poisson bracket. The quantization of an integrable system in this context is such a quantization of a Poisson algebra that $q(\Im)$ is a commutative subalgebra (not only at the first order in $\hbar$) in $A(M)$. The importance of this condition is obvious, as like as in the classical case the dynamics can be restricted to the common level of integrals, the quantum problem of finding the eigenvectors can be restricted to the common eigenspace of quantum integrals.

The main example of this paper is the quantization of the space g^* for a Lie algebra $g = \mathfrak{gl}_n$ with the Kirillov-Kostant Poisson bracket. The space of functions in this case is just the symmetric algebra $C(M) = S^*(g)$. The quantum algebra is the universal enveloping algebra $U(g)$ deformed in a standard manner: one takes new generators $\widetilde{e}_{ij} = \hbar e_{ij}$ than the defining relations in $U(g)[\hbar]$ became

$$[\widetilde{e}_{ij}, \widetilde{e}_{kl}] = \hbar(\delta_{jk}\widetilde{e}_{il} - \delta_{li}\widetilde{e}_{kj}).$$

The quantization morphism on homogeneous polynomials is given by fixing some normal ordering. The classical limit consists in tending $\hbar \mapsto 0$.

The same procedure can be realized without thinking of formal parameter due to the structure of filtered algebra on $U(\mathfrak{gl}_n)$. Indeed, the quantization can be considered as a map from $S^*(\mathfrak{gl}_n)$ to just $U(\mathfrak{gl}_n)$ given by the same normal ordering. Let us note that the choice of such an ordering does not affect the result of passing to the associated graded algebra $Gr(U(\mathfrak{gl}_n)) = S^*(\mathfrak{gl}_n)$. Hence one can consider the factorization morphism $\rho : U(\mathfrak{gl}_n) \to Gr(U(\mathfrak{gl}_n))$ as a classical limit. Indeed, the composition of quantization and factorization is an identity on homogeneous element of the symmetric algebra $S^*(\mathfrak{gl}_n)$.

Gaudin system. Let us consider the direct sum of k coadjoint $\mathfrak{gl}_n$-orbits $\oplus_i \mathcal{O}_i$ with an induced Poisson (symplectic) structure and the diagonal symplectic GL_n-action. An element of this space can be represented by the Lax operator

$$L(u) = \sum_i \frac{\Phi_i}{u - z_i} \tag{1}$$

with $\Phi_i \in \mathcal{O}_i$. Fixing a basis one can think of Φ_i as a matrix valued function on $\mathcal{O}_i$ with matrix elements f_{kl} - generators of $\mathfrak{gl}_n$ considered as linear functions on the i-th orbit. Then the functions

$$I_i(u) = TrL^i(u) \quad \text{commute} \quad \{I_i(u), I_j(v)\} = 0$$

and are invariant under the diagonal GL_n-action. The images of these functions on the symplectic reduction space $M = \oplus_i \mathcal{O}_i // GL_n$ give the integrable system. The proof of the integrability can be found for example in [1] and is based on the r-matrix representation of the Poisson bracket:

$$\{L(u) \otimes L(v)\} = [r(u-v), L(u) \otimes 1 + 1 \otimes L(v)] \quad \text{with} \quad r(u) = \frac{P}{u}$$

where P is a permutation operator in the tensor product. There exist several partial responses to the quantization problem of this integrable system. For example in the $\mathfrak{gl}_2$ case just the same expressions

$$I_2(u) = TrL^2(u)$$

with f_{kl} changed to the $U(\mathfrak{gl}_2)$ generators e_{kl} give the commutative family of elements in $U(\mathfrak{gl}_2)^{\otimes k}$. The attempt to generalize this result to $\mathfrak{gl}_n$ in [2] was only partially successful. There was obtained that quantum expressions for $I_2(u), I_3(u)$ commute but it is not true in general for the higher integrals.

Main result. Consider the quantum Lax operator given by the expression (1) where Φ_i are matrices with elements in $U(\mathfrak{gl}_n)^{\otimes k}$ such that $(\Phi_i)_{kl}$ is the e_{kl} generator of $\mathfrak{gl}_n$ lying in the i-component of the tensor product. Consider the following differential operators applied to the function 1:

$$QI_i(u) = Tr_{1,\ldots,n} A_n (L_1(u) - \partial_u)(L_2(u) - \partial_u) \ldots (L_i(u) - \partial_u)\mathbf{1} \tag{2}$$

where A_k is the antisymmetrizer in $Mat_{n\times n}^{\otimes n}$.

Main Theorem. *The quantities $QI_i(u)$ constitute a commutative family in $U(\mathfrak{gl}_n)^{\otimes k}(u)$ in the following sense*

$$[QI_i(u), QI_j(v)] = 0, \qquad \forall i, j = 1, \ldots n, \ \forall u, v;$$

and their classical limit in the sense of associated graded algebra coincides with the family of classical Gaudin hamiltonians.

Plan of the paper. In section 2.1 we recall the construction of the Yangian $Y(\mathfrak{gl}_n)$ and the Bethe subalgebra in it, which is a maximal commutative subalgebra for general choice of parameters. Further, considering the k-th tensor power of the evaluation homomorphism we obtain a commutative family in $U(\mathfrak{gl}_n)^{\otimes k}$. In section 2.2 we extend the Bethe subalgebra to the commutative family of formal expressions in h. This formal family gives the standard Bethe subalgebra at $\mathsf{h} = 1$, whereas we use its special limit at $\mathsf{h} = 0$. We also slightly change a basis using only linear transformations. The main result of the paper is contained in section 2.3.

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2 Quantum Gaudin system

2.1 Bethe subalgebra

The Yangian was introduced by Drinfeld [3] and take an important role in describing rational solutions of the Yang-Baxter Equation. $Y(\mathfrak{gl}_n)$ is generated by elements $t_{ij}^{(k)}$ subject to specific relations on the generating function $T(u)$ with values in $Y(\mathfrak{gl}_n) \otimes End(\mathbb{C}^n)$

$$T(u) = \sum_{i,j} E_{ij} \otimes t_{ij}(u), \qquad t_{ij}(u) = \delta_{ij} + \sum_k t_{ij}^{(k)} u^{-k},$$

where E_{ij} are matrices with 1 on the i,j-th place. The relations on this generating function involve the Yang R-matrix

$$R(u) = 1 - \frac{1}{u} \sum_{i,j} E_{ij} \otimes E_{ji}$$

and reads as the following equation

$$R(z-u) T_1(z) T_2(u) = T_2(u) T_1(z) R(z-u)$$

in $End(\mathbb{C}^n)^{\otimes 2} \otimes Y(\mathfrak{gl}_n)[z,u]$, where

$$T_1(z) = \sum_{i,j} E_{ij} \otimes 1 \otimes t_{ij}(z), \quad T_2(u) = \sum_{i,j} 1 \otimes E_{ij} \otimes t_{ij}(u).$$

There is a known realization of $Y(\mathfrak{gl}_n)$ in $U(\mathfrak{gl}_n)$ i.e. a homomorphism $\rho_1 : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$

$$T(u) = 1 + \frac{1}{u} \sum_{i,j} E_{ij} \otimes e_{ij} \stackrel{def}{=}: 1 + \frac{\Phi}{u}, \tag{3}$$

where e_{ij} are generators of $\mathfrak{gl}_n$. The following expression also provide a realization of Yangian in $U(\mathfrak{gl}_n)^{\otimes k}$ for the given k-tuple of complex numbers $\alpha = (z_1, \ldots, z_k)$

$$T^\alpha(u) = T^1(u - z_1) T^2(u - z_2) \ldots T^k(u - z_k),$$

where $T^l(u-z_l)$ is the realization given by (3) with e_{ij} lying in the l-th component of $U(\mathfrak{gl}_n)^{\otimes k}$. Let ρ_α be the corresponding mapping $\rho_\alpha : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes k}$.

Lemma 1 *ρ_α is a homomorphism. This is equivalent to the relation*

$$R(z-u)T_1^\alpha(z)T_2^\alpha(u) = T_2^\alpha(u)T_1^\alpha(z)R(z-u)$$

in $End(\mathbb{C}^n)^{\otimes 2} \otimes U(\mathfrak{gl}_n)^{\otimes k}[z,u]$.
Proof. Here we use the fact that $T_1^l(z)$ commutes with $T_2^k(u)$ if $l \neq k$. Indeed,

$$R(z-u)T_1^1(z-z_1)T_1^2(z-z_2)\dots T_1^k(z-z_k)T_2^1(u-z_1)T_2^2(u-z_2)\dots T_2^k(u-z_k) =$$

$$R(z-u)T_1^1(z-z_1)T_2^1(u-z_1)T_1^2(z-z_2)\dots T_1^k(z-z_k)T_2^2(u-z_2)\dots T_2^k(u-z_k) =$$
$$T_2^1(u-z_1)T_1^1(z-z_1)R(z-u)T_1^2(z-z_2)\dots T_1^k(z-z_k)T_2^2(u-z_2)\dots T_2^k(u-z_k) =$$

et cetera...
□

The simple way to obtain the commuting family in $U(\mathfrak{gl}_n)^{\otimes k}$ is to take the image $\rho_\alpha(Z(Y(\mathfrak{gl}_n)))$ of the center of Yangian. The homomorphism ρ_α is not surjective and that is why this commuting family is not trivial (contains not only constants). The center of Yangian is given by

$$qdetT^\alpha(u) = \sum_{\sigma \in S_n} sgn(\sigma) t^\alpha_{\sigma(1),1}(u)\dots t^\alpha_{\sigma(n),n}(u-n+1)$$

see for example Proposition 2.7 from [4]. Unfortunately this family is too small, $T^\alpha(u)$ is a rational function on u with k simple poles, $qdetT^\alpha(u)$ has kn simple poles, the corresponding residues play the role of formal quantum hamiltonians whereas the dimension of the classical phase space is of order kn^2. There is another construction enlarging the family of central elements in Yangian up to a maximal commutative subalgebra (see [5],[6] or 2.14 of [7]): let C be an $n \times n$ matrix, $T(u)$ the generating function for the elements of $Y(\mathfrak{gl}_n)$, A_n is the matrix of antisymmetrizer in $\mathbb{C}^{n \otimes n}$ and

$$T_i(u) = \sum_{kl} 1 \otimes \dots \otimes 1 \otimes \overset{i}{E_{kl}} \otimes 1 \otimes \dots \otimes 1 \otimes t_{kl}(u)$$

then the coefficients of

$$\tau_i(u) = TrA_nT_1(u)T_2(u-1)\dots T_i(u-i+1)C_{i+1}\dots C_n \quad i = 1, \dots n$$

generate the commuting family
$$[\tau_i(u), \tau_j(v)] = 0$$
which is maximal if the matrix C has simple spectra. In the realization $T^\alpha(u)$ each $\tau_i(u)$ is a rational function on u with ik simple poles, the total number of formal quantum conserved quantities is equal to $k\frac{(n+1)n}{2}$ which is approximatively half the dimension of the classical phase space.

2.2 Formal deformation

Let us consider the canonical rescaling of the universal enveloping algebra obtained by multiplying structure constants by h. The corresponding $\mathfrak{gl}_n$ representation of Yangian $T(u)$ looks as follows

$$T(u) = 1 + \frac{\mathsf{h}\Phi}{u}.$$

It solves YBE with the rescaled Yang R-matrix

$$R(u) = 1 - \frac{\mathsf{h}P}{u}.$$

The construction of the commuting family is reproduced literally in this case:

$$\boxed{\tau_i(u) = (TrA_n T_1^\alpha(u) T_2^\alpha(u-\mathsf{h}) \dots T_i^\alpha(u-(i-1)\mathsf{h}) C_{i+1} \dots C_n)} \quad i = 1, \dots n. \tag{4}$$

We will use later the following obvious

Lemma 1 *Let two formal in* h *expressions*

$$M(\mathsf{h}) = \sum_{i=m}^{\infty} M_i \mathsf{h}^i \quad and \quad N(\mathsf{h}) = \sum_{i=n}^{\infty} N_i \mathsf{h}^i$$

with values in some associative algebra $\mathcal{A}$ *commute, then their first coefficients also commute*

$$[M_m, N_n] = 0.$$

It is also true for the first non-central coefficients of formal series.

Remark 1 *Let us note that the classical Gaudin Lax operator can be recovered as a classical limit of* T^α :

$$T^\alpha(u) = 1 + \mathsf{h}L(u) + O(\mathsf{h}^2)$$

where

$$L(u) = \sum_i \frac{\Phi_i}{u - z_i}$$

is the first non-central coefficient of the operator-valued matrix. This expression lies in the first filtration component and after taking classical limit remains of the same form (with described in Introduction exchange of the meaning of matrix elements).

2.3 Higher Hamiltonians

Let us introduce "quasiclassic" hamiltonians:

$$\boxed{I_k = TrA_n L_1(u) L_2(u) \dots L_k(u).}$$

Their classical limits in the sense of "Gr" are given by the following expressions:

$$\begin{aligned} i_1 &= TrL(u)(n-1)!; \\ i_2 &= (Tr^2 L(u) - TrL^2(u))(n-2)!; \\ i_3 &= (2TrL^3(u) - 3TrL(u)TrL^2(u) + Tr^3 L(u))(n-3)! \end{aligned} \tag{5}$$

and can be taken as a basis of classical Gaudin hamiltonians.

Consider formal expressions

$$\boxed{s_i = \sum_{j=0}^{i} (-1)^j C_i^j \tau_{i-j}} \tag{6}$$

where $\tau_0 = TrA_n 1 = n!$.

Lemma 2 *s_i has the following expansion on* h

$$s_i(u) = \mathsf{h}^i QI_i(u) + O(\mathsf{h}^{i+1})$$

where QI_i is given by the formula

$$\boxed{QI_i(u) = Tr_{1,\ldots,n} A_n (L_1(u) - \partial_u)(L_2(u) - \partial_u)\ldots(L_i(u) - \partial_u)\mathbf{1}} \tag{7}$$

where A_i is the antisymmetrizer matrix in $Mat_{n\times n}^{\otimes i}$ and $L_j(u)$ is the quantum Lax operator (1) lying in the j-th tensor component in $Mat_{n\times n}^{\otimes i}$.

Proof. Let us represent the formula for the generators of the Bethe subalgebra (4) in a quite different manner:

$$\tau_i(u-\mathsf{h}) e^{-i\mathsf{h}\partial_u} = TrA_n e^{-\mathsf{h}\partial_u} T_1(u) e^{-\mathsf{h}\partial_u} T_2(u) \ldots e^{-\mathsf{h}\partial_u} T_i(u) \tag{8}$$

where expressions on both sides are differential operators on u with values in $U(\mathfrak{gl}_n)^{\otimes k}(u)$. These formulas can be combined in a sort of a generating function:

$$\begin{aligned} & TrA_n (e^{-\mathsf{h}\partial_u} T_1(u) - 1)(e^{-\mathsf{h}\partial_u} T_2(u) - 1)\ldots(e^{-\mathsf{h}\partial_u} T_i(u) - 1) \\ = & \sum_{j=0}^{i} \tau_j(u-\mathsf{h})(-1)^{i-j} C_i^j e^{(j-i)\mathsf{h}\partial_u} \end{aligned} \tag{9}$$

Applying then both sides to the function 1 and using the fact that

$$e^{-\mathsf{h}\partial_u} T(u) - 1 = \mathsf{h}(L(u) - \partial_u) + O(\mathsf{h}^2)$$

we obtain

$$s_i(u) = \mathsf{h}^i TrA_n (L_1(u) - \partial_u)(L_2(u) - \partial_u)\ldots(L_i(u) - \partial_u)\mathbf{1} + O(\mathsf{h}^{i+1}).$$

□

Hence we have obtained that QI_i commute. Indeed, $[s_i(u), s_j(v)] = 0$ due to the fact that $s_i(u)$ are linear combinations of generators of the Bethe subalgebra. Then, $[QI_i(u), QI_j(v)] = 0$ in virtue of lemma 1. It is quite evident that the classical limit of QI_i coincides with the classic hamiltonian, the element of the highest order in (7) is

$$TrA_n L_1(u) L_2(u) \ldots L_i(u).$$

Thus we proved the main theorem of this paper:

Theorem 1 *The quantities $QI_i(u)$ provide a quantization of the classical Gaudin model.*

Let us analyze the obtained quantum hamiltonians at small order:

- $\mathsf{l}=1$
$$s_1 = \mathsf{h} TrL(u)(n-1)! + O(\mathsf{h}^2);$$
the first term at h is just the "semiclassic" hamiltonian $I_1(u)$.

- $\mathsf{l}=2$
$$s_2 = \tau_2 - 2\tau_1 + \tau_0 = \mathsf{h}^2(-\partial_u TrL(u)(n-1)! + TrA_n L_1(u)L_2(u)) + O(\mathsf{h}^3);$$
the first term at h^2 can be expressed as
$$I_2(u) - \partial_u I_1(u).$$

- $\mathsf{l}=3$
$$\begin{aligned} s_3 &= \mathsf{h}^3 TrA_n(\partial_u^2 L_1(u) - \partial_u L_1(u)L_2(u) - 2L_1(u)\partial_u L_2(u) \\ &+ L_1(u)L_2(u)L_3(u)) + O(\mathsf{h}^4). \end{aligned}$$
The first term at h^3 can be simplified as follows:
$$I_3(u) - \frac{3}{2}\partial_u I_2(u) + \partial_u^2 I_1(u);$$
where we have used the property $TrA_n\partial_u L_1(u)L_2(u) = TrA_n L_1(u)\partial_u L_2(u)$ which is not true for the number of multipliers greater then 2.

- $\mathsf{l}=4$
$$\begin{aligned} s_4 &= \mathsf{h}^4 TrA_n(-\partial_u^3 L_1(u) + \partial_u^2 L_1(u)L_2(u) + 3L_1(u)\partial_u^2 L_2(u) + 3\partial_u L_1(u)\partial_u L_2(u) \\ &- \partial_u L_1(u)L_2(u)L_3(u) - 2L_1(u)\partial_u L_2(u)L_3(u) - 3L_1(u)L_2(u)\partial_u L_3(u) \\ &+ L_1(u)L_2(u)L_3(u))L_4(u)) + O(\mathsf{h}^5). \end{aligned}$$
The first term here at h^4 also can be simplified:
$$\begin{aligned} &I_4(u) - \partial_u I_3(u) + 2\partial_u^2 I_2(u) - \partial_u^3 I_1(u) \\ -\ &TrA_n(\partial_u L_1(u)\partial_u L_2(u) + L_1(u)\partial_u L_2(u)L_3(u) + 2L_1(u)L_2(u)\partial_u L_3(u)) \end{aligned}$$

The arguments of the previous section give us that the first coefficients as like as their derivatives commute and we obtain that the quasiclassic hamiltonians for $i = 1, \ldots, 3$ lie in our commutative family. However the higher hamiltonians $i \geq 4$ have unavoidable quantum corrections.

Concluding remarks

Pull back to $Y(\mathfrak{gl}_n)$ The proposed construction for quantum hamiltonians can be realized in a more general setup, it can be pulled back to the Yangian. In this case it provides a commutative family expressed only in the first order generators of the Yangian

$$QI_i(u) = Tr_{1,\dots,n} A_n (T_1^{(1)}(u) - \partial_u)(T_2^{(1)}(u) - \partial_u) \dots (T_i^{(1)}(u) - \partial_u)\mathbf{1}.$$

In this way one could expect to quantize a quite wider class of models: rational matrices with higher order poles and the Hitchin system on singular curves.

Bethe ansatz The obtained formulas for quantum hamiltonians provide a kind of decomposition which could be tractable in the context of the Bethe ansatz for higher dimensional spin components of the Gaudin model.

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